New self-dual solutions of $SU(2)$ Yang-Mills theory in Euclidean Schwarzschild space

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We present a systematic study of spherically symmetric self-dual solutions of SU(2) Yang-Mills theory on Euclidean Schwarzschild space. All the previously known solutions are recovered and a new one-parameter family of instantons is obtained. The newly found solutions have continuous actions and interpolate between the classic Charap and Duff instantons. We examine the physical properties of this family and show that it consists of dyons of unit (magnetic and electric) charge.

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I. INTRODUCTION

The study of SU(2) instantons, i.e., finite action smooth (anti) self-dual solutions of the SU(2) Yang-Mills (YM) equation, has been extensively carried out in Euclidean flat space, and numerous interesting properties of them are by now well established [1]. These solutions correspond to global minima of the action in each topological sector and, therefore, are associated with the leading terms in semiclassical approximations to the path integral. In particular, instantons are associated with tunneling between degenerate vacua in Yang-Mills theory at zero temperature.

At the classical level, it is well known that these solutions are classified by $\pi_3(S^3) = \mathbb{Z}$ and that their action, which coincides with the first Pontryagin class, is restricted to integer values [2]. These results are related to the topology of flat space and are not expected to hold in manifolds with other topologies.

One important feature of self-dual solutions in Euclidean backgrounds (not only in flat space) is that the associated energy-momentum tensor vanishes and thus such field configurations do not disturb the geometry of spacetime [3]. Hence, given a solution of Einstein's equation in vacuum and a self-dual solution of YM theory in such background, we immediately have a full solution of Einstein-Yang-Mills theory. Therefore, solutions of Einstein's equation in vacuum provide the most interesting manifolds (from a physical viewpoint) to study self-dual solutions of YM theory.

Furthermore, although gravity is expected to have a weak effect on the perturbative sector of field theories, it might drastically change the nonperturbative sector. This is not a surprise since gravity can change the topology of spacetime and therefore strongly affect the configuration space for the fields. For example, a number of works have considered self-dual solutions of Yang-Mills theory in various Euclidean backgrounds and found solutions with no counterpart in flat space [3, 4, 5, 6, 7, 8, 9]. Even the usual tunneling interpretation does not hold in some spacetime geometries [10]. Another remarkable difference is that the action can take continuous values as has been exhibited for the AdS, Schwarzschild and other spherically symmetric spacetimes [6, 8].

In this work we approach the question of determining, in a systematic way, the possible spherically symmetric smooth self-dual solutions of Yang-Mills theory in the Euclidean Schwarzschild background. As a result, we recover all the previously known solutions and, moreover, disclose a whole new one-parameter family of instantons with continuous actions. This family interpolates between the classic Charap and Duff's solutions [3, 4] with actions 1 and 2. We examine the physical properties of this family and show that it consists of dyons of unit (magnetic and electric) charge. Like the previously obtained instantons in this background, these new solutions are associated with a constant gauge-invariant static potential (no tunneling).

The structure of this paper is as follows. We first discuss the spherically symmetric ansatz employed by us, paying special attention to regularity issues that naturally arise in this context. We then prove the existence of the aforementioned solutions, study their global behavior and examine their physical properties. Finally, we employ the recently introduced mapping method [11, 12] to obtain a new variable according to which these solutions can be analytically expressed as a power series with apparently infinite radius of convergence. We close with some general remarks which include a brief discussion on the possible mathematical relevance of our results to the study of instantons in asymptotically locally flat (ALF) geometries [13].

II. ANSATZ

The Euclidean Schwarzschild space is a noncompact complete Riemannian manifold topologically given by $M = S^2 \times \mathbb{R}^2$, and endowed with the following metric, defined on the open set $S^2 \times (\mathbb{R}^2 \backslash \{0\}) \subset M$:

$$ds^2 = \left(1 - \frac{2m}{r}\right) d\tau^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \qquad (1)$$

In the above expression $m > 0$, $d\Omega^2$ is the usual round metric in S^2, and (r, τ) are polar coordinates on $\mathbb{R}^2 \backslash \{0\}$, which take values on the intervals $(2m, \infty)$ and $[0, 8\pi m)$, respectively. It is well known that the metric (1) can be uniquely extended to the whole M, and that such extension is a nonsingular solution of the vacuum Einstein equations, with nontrivial Euler characteristic $\chi = 2$.

Using coordinates $x^\mu = (\tau, r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta)$, with (θ, φ) the usual angular coordinates in S^2, we will adopt the general spherically symmetric ansatz for the gauge potential [10, 14]:

$$A = \frac{\sigma_a}{2i} \left[\phi(\tau, r) \frac{x^a}{r} d\tau + \beta(\tau, r) \frac{x^a x^c}{r^2} dx^c + (\gamma(\tau, r) - 1)\epsilon_{abc} \frac{x^b}{r} d\left(\frac{x^c}{r}\right) + \alpha(\tau, r) d\left(\frac{x^a}{r}\right) \right], \qquad (2)$$

where σ_a are the Pauli matrices and ϵ_{abc} is the totally antisymmetric tensor, with $\epsilon_{123} = +1$.

Because of the angular character of the time coordinate, care must be taken with the boundary conditions to ensure regularity of the gauge potential. Clearly all of the ansatz functions must be periodic in time. Also, as $r \to 2m$, ϕ must vanish and all the other functions must become time-independent.

Under a gauge transformation $\tilde{A} = U^{-1}AU + U^{-1}dU$, with $U = \exp(-if(\tau, r)x^a\sigma_a/2r)$, the form of the ansatz is unchanged (this corresponds to a $U(1)$-symmetry of the problem [10, 14]) and this can be used to choose a gauge in which $\beta = 0$. If we are interested in time-independent self-dual solutions [15] we can also set $\gamma = 0$. The (singular) gauge transformation given by $U = \exp(-i\varphi\sigma_3/2)\exp(-i\theta\sigma_2/2)$ then leads to the ansatz [6]

$$A = \alpha(r) d\theta \frac{\sigma_1}{2i} + \alpha(r) \sin\theta \, d\varphi \frac{\sigma_2}{2i} + (cos\theta \, d\varphi + \phi(r) \, d\tau) \frac{\sigma_3}{2i}. \qquad (3)$$

Note that given a singular potential in the form of Eq. (3) one can turn it back to the form of Eq. (2) and get a nonsingular potential as long as we ensure that the regularity constraints are obeyed.

For a potential given by the above expression the self-duality equations are:

$$\alpha'(r) = -\left(1 - \frac{2m}{r}\right)^{-1} \alpha(r)\,\phi(r), \tag{4a}$$

$$\phi'(r) = \frac{1 - \alpha^2(r)}{r^2}. \tag{4b}$$

It immediately follows that ϕ is governed by the ordinary differential equation

$$\frac{r}{2}(r - 2m)\frac{d^2\phi}{dr^2} + (r - 2m)\frac{d\phi}{dr} - \phi + r^2\frac{d\phi}{dr}\phi = 0, \tag{5}$$

and that α is given, in terms of ϕ, by

$$\alpha^2(r) = 1 - r^2\phi'(r). \tag{6}$$

Notice that this imposes a restriction on the solutions of Eq. (5), since α needs to be a real function ($\alpha^2(r) \geq 0$) in order that A be $\mathfrak{su}(2)$-valued [16].

The Lagrangian associated with these solutions is easily calculated,

$$\frac{1}{8\pi^2}\,\text{tr}(F \wedge \star F) = -\frac{1}{8\pi^2}\frac{\partial}{\partial r}\left[(1 - \alpha^2)\phi\right]\,d\tau \wedge dr \wedge d\Omega, \tag{7}$$

and using this expression one finds that the action is given by

$$S = -\frac{1}{8\pi^2}\int_M \text{tr}(F \wedge \star F) = 4m\left[(1 - \alpha^2)\phi\right]\Big|_{2m}^{\infty}. \tag{8}$$

The solutions we will be interested in satisfy $\alpha(2m) = 0$ and $\alpha(r) \to 0$ as r goes to infinity (see section III). In this case the action depends only on the asymptotic values of $\phi(r)$:

$$S = 4m\left[\phi(\infty) - \phi(2m)\right]. \tag{9}$$

We note that this expression is not gauge invariant: its simple form is a direct consequence of the gauge choice associated with ansatz (3).

Following the approach described in [18] (see also [10, 19]) one can define an Abelian field strength given by

$$f_{\mu\nu} = -i\,\text{tr}\left[\sigma_3\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\right].$$

This can be identified with an electromagnetic field and, in particular we find that

$$f^{0i} = \phi'(r)\frac{x^i}{r}, \tag{10a}$$

$$\star f^{0i} = \frac{x^i}{r^3}. \tag{10b}$$

Therefore it is clear that all configurations given by ansatz (3) have unit magnetic charge. The electric charge depends on the asymptotic behavior of $\phi(r)$.

III. SOLUTIONS

A. General discussion

Three solutions immediately arise from Eqs. (4), which are the classic Charap and Duff solutions [3]:

i. $\phi(r) = 0$, $\alpha(r) = 1$. In this case, $S = 0$.

ii. $\phi(r) = -\frac{m}{r^2}$, $\alpha^2(r) = 1 - \frac{2m}{r}$. In this case, $S = 1$.

iii. $\phi(r) = c - \frac{1}{r}$, where c is a constant, and $\alpha(r) = 0$. In this case, $S = 2$.

We see from Eqs. (10) that the solutions with actions 1 and 2 above correspond to a monopole and a dyon, respectively [10]. The third solution is moreover Abelian, since it lies in a fixed $\mathfrak{u}(1)$ inside $\mathfrak{su}(2)$ as can be seen from Eq. (3) (see also [20]).

It is evident that the second and third (for general c) solutions above do not satisfy the regularity condition $\phi(2m) = 0$. Going back to Eq. (2) we can see that given a gauge potential with $\phi(2m) = \lambda \neq 0$ we can turn it into a regular potential by performing a singular gauge transformation with $U = \exp(i\lambda t\, x^a \sigma_a/2r)$, as long as $8\pi m\lambda = 2\pi p$, with $p \in \mathbb{Z}$, and $\alpha(2m) = 0$. This is clearly the case for the aforementioned solutions if we choose c properly. Notice that, unless $\alpha(r) = 0$, the nonsingular solution will be time-dependent.

A straightforward application of the power series method and the aforementioned condition on $\phi(2m) = \frac{p}{4m}$ restrict the possible solutions of (5), in a neighborhood of $2m$, to $\phi(r) = \sum_{k=0}^{\infty} a_k(r - 2m)^k$, with $a_0 = \frac{p}{4m}, p \in \mathbb{Z}$. It turns out that only the Abelian solution is possible for $p > 0$. For $p \leq 0$ a much more interesting situation arises. In this case, the coefficient a_{-p+1} is not fixed by the condition $a_0 = \frac{p}{4m}$, and the remaining coefficients a_k are determined by a complicated nonlinear recurrence relation that depends on all the coefficients with index less than k (and hence on a_{-p+1}). As a result, for each $p \leq 0$, the free parameter a_{-p+1} gives rise to a whole family of solutions of (5).

We have strong numerical evidence that, for $p \leq -2$, the only possible solution with finite action is again the Abelian solution: all other possible solutions of Eq. (5) either diverge as $r \to \infty$ or render α imaginary. In this way, we turn our attention to the two remaining possibilities, namely $p = -1$ and $p = 0$. These two cases have solutions with very dissimilar behaviors. In fact, as $r \to 2m$, $\alpha(r)$ goes like $(r - 2m)^{1/2}$ and $const + (r - 2m)$ for $p = -1$ and $p = 0$, respectively. More to the point, the solutions associated with $p = -1$ and $p = 0$ are not gauge equivalent (except for the trivial Abelian case). This can be easily seen by plugging the power series expressions for these solutions into the gauge-invariant expression (7). It is not difficult to show that the case $p = 0$ exactly corresponds to the results obtained in [6], in which the authors present a family of solutions with action ranging from 0 to 2. Therefore, in this work we focus on the remaining possible choice ($p = -1$) which, as discussed above, leads to time-dependent solutions in the appropriate nonsingular gauge.

It follows from the series solution that $a_1 = \phi'(2m) = \frac{1}{4m^2}$. Writing $s = r - 2m$, the first few terms of the corresponding solution are given by

$$\phi_\kappa(s + 2m) = -\frac{1}{4m} + \frac{s}{4m^2} + \frac{\kappa}{16m^3}s^2 - \frac{\kappa + 1}{16m^4}s^3 - \frac{\kappa^2 - 7\kappa - 10}{256m^5}s^4 + \mathcal{O}(s^5), \qquad (11)$$

where we introduced the free dimensionless parameter κ given by $\kappa = 16m^3\, a_2$. It is not hard to show, directly from the recurrence relation satisfied by a_n, that:

1. for $\kappa = -3$, ϕ_κ reduces to $\phi_{-3}(r) = -\frac{m}{r^2}$, which is the aforementioned Charap and Duff's solution with $S = 1$;

2. for $\kappa = -2$, ϕ_κ reduces to $\phi_{-2}(r) = \frac{1}{4m} - \frac{1}{r}$, which is the aforementioned Charap and Duff's solution with $S = 2$.

Unfortunately, the radius of convergence of (11) is, in general, not greater than $2m$. In this way, this method fails to provide any insight about $\phi(r)$ for $r \gg 1$. In particular, it yields no information whatsoever on the finiteness (not to say the value) of the action associated with each ϕ_κ (Eq. (9)). This kind of question requires considerations on the global behavior of these solutions and will be elucidated in the last part of this section by studying some analytic properties shared by them. Then, in section IV, we will use a mapping method to obtain a series solution that has an infinite radius of convergence.

Despite its aforementioned limitations, Eq. (11) can be used to generate initial values $(\phi_\kappa(2m + \epsilon), \phi'_\kappa(2m + \epsilon))$ for (5), with ϵ positive and arbitrarily small. Such initial values can then be employed to numerically integrate (5) [21]. Figure 1 shows representative solutions obtained in this way. The thick curves correspond to the classic Charap and Duff solutions [3].
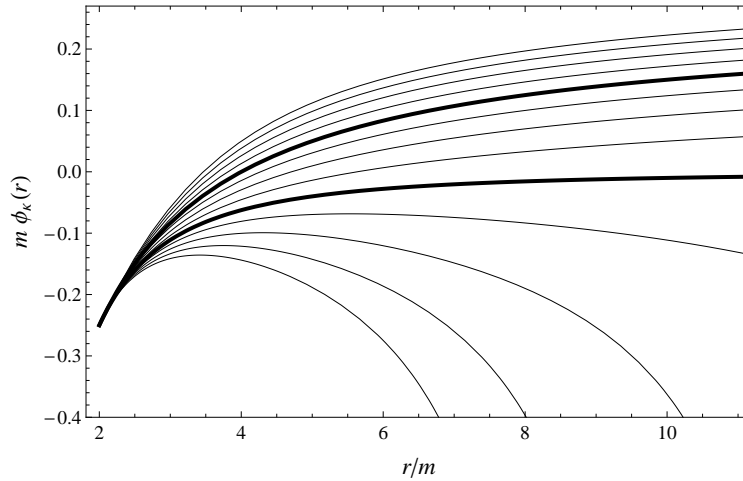

FIG. 1: Plots of $\phi_\kappa(r)$ for evenly spaced values of κ going from -4 (lowest curve) to -1 (uppermost curve). The thick curves correspond to the Charap and Duff solutions ($\kappa = -3$ for the lower thick curve and $\kappa = -2$ for the upper thick curve).

Figure 1 suggests that:

1. solutions with $\kappa < -3$ (below the lower thick curve) are not limited, and therefore do not have finite action (cf Eq. (9));

2. solutions with $-3 < \kappa < -2$ (between the thick curves) interpolate between the Charap e Duff classic solutions;

3. solutions with different values of κ do not intersect.

In what follows, we show that the solutions of interest are indeed those with $-3 \leq \kappa \leq 2$. In this case, the corresponding ϕ_κ does give rise to pairs (ϕ, α) (cf Eq. (3)) associated with solutions that interpolate between Charap and Duff's instantons with actions 1 and 2. We also show that item 3 above is true as long as $\kappa \leq -2$, which is precisely the condition for $\alpha(r)$, as defined by Eq. (6), to be real-valued.

B. Global behavior of the solutions

We start by noting a useful local property of the solutions ϕ_κ. Let $\kappa_1, \kappa_2 \in \mathbb{R}$. If $\kappa_1 < \kappa_2$, then, for $r - 2m \ll 1$:

$$\phi_{\kappa_1}(r) < \phi_{\kappa_2}(r), \tag{12a}$$

$$\phi'_{\kappa_1}(r) < \phi'_{\kappa_2}(r), \tag{12b}$$

which follows directly from Eq. (11). Therefore plots of $\phi_\kappa(r)$ (as in Fig. 1) corresponding to different values of κ do not intersect for r sufficiently close to $2m$. A careful analysis of the ordinary differential equation (ODE) (5) leads to a much stronger statement. In fact, we show in the Appendix that for $\kappa_1 < \kappa_2 \leq -2$:

$$\phi_{\kappa_1}(r) < \phi_{\kappa_2}(r) \quad \forall r \in (2m, \infty), \tag{13a}$$

$$\phi'_{\kappa_1}(r) < \phi'_{\kappa_2}(r) \quad \forall r \in (2m, \infty). \tag{13b}$$

It follows from Eq. (13a) that plots of $\phi_\kappa(r)$ corresponding to different values of κ do not intersect for any $r \in (2m, \infty)$ as long as $\kappa \leq -2$ [22]. We also note that taking $\kappa_2 = -2$ in Eq. (13b) leads to $\phi'_\kappa(r) \leq 1/r^2$ whenever $\kappa < -2$, which is precisely the condition needed to guarantee that α is real.

Consider now a fixed value of $\kappa \in (-3, -2)$. A direct consequence of the above discussion is that the associated action S_κ is finite in this case. Indeed, using Eq. (13b) with $\kappa_1 = -3$, we see that $\frac{2m}{r^3} < \phi'_\kappa(r)$, and hence ϕ_κ is a strictly increasing function. Moreover, it follows directly from Eq. (13a) that

$$-\frac{m}{r^2} < \phi_\kappa(r) < \frac{1}{4m} - \frac{1}{r}, \qquad \text{for } -3 < \kappa < -2,$$

and thus $\phi_\kappa(r)$ is limited. Therefore, the limit

$$C_\kappa := \lim_{r \to \infty} \phi_\kappa(r) \tag{14}$$

exists and $C_\kappa \in (0, \frac{1}{4m}]$ for $-3 < \kappa \leq -2$. On the other hand, it is easy to show, directly from Eq. (5), that the asymptotic behavior of $\phi_\kappa(r)$, also for $-3 < \kappa \leq -2$, is given by

$$\phi_\kappa(r) \to C_\kappa - \frac{1}{r} + \lambda \frac{e^{-2C_\kappa r}}{2C_\kappa} \qquad (r \gg 1) \tag{15}$$

for some $\lambda \in \mathbb{R}$, which implies $\alpha \to 0$. This shows that the action can be calculated from Eq. (9) and that it is indeed finite. It is also clear from Eqs. (10a) and (15) that these solutions have unit electric charge and therefore correspond to dyons.

Figure 2 shows a representative subset of the family of solutions ϕ_κ, with $\kappa \in [-3, -2]$ (these plots were generated as those in Fig. 1). Notice that it numerically illustrates all the analytical results obtained above.

It follows from Eq. (14) that, given $R > 2m$, $\frac{m}{R^2} < \phi_\kappa(\infty) - \phi_\kappa(R) < \frac{1}{R}$. Thus

$$\phi_\kappa(R) + \frac{m}{R^2} < \phi_\kappa(\infty) < \phi_\kappa(R) + \frac{1}{R}$$

and this leads to accurate estimates for the action via Eq. (9):
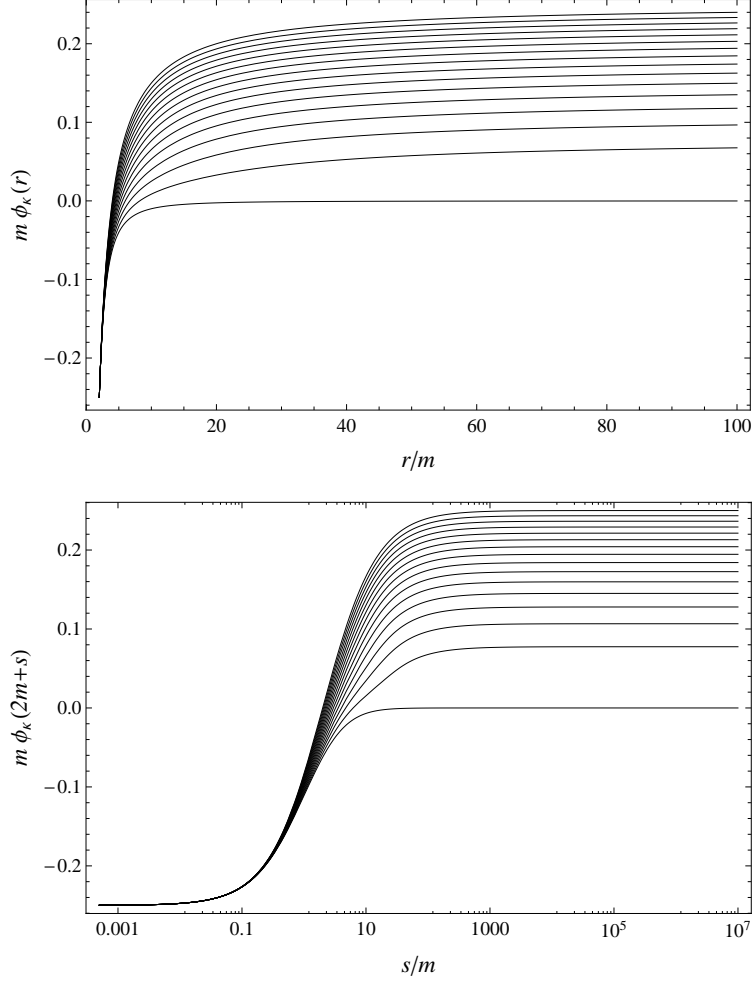
$$S_\kappa = 1 + 4m\,\phi(\infty).$$

FIG. 2: Top: Plots of $\phi_\kappa(r)$ as a function of r for evenly spaced values of κ going from $\kappa = -3$ (lowest curve) to $\kappa = -2$ (uppermost curve). Bottom: Semilog plots of $\phi_\kappa(s + 2m)$ as a function of s associated with the same values of κ as before.
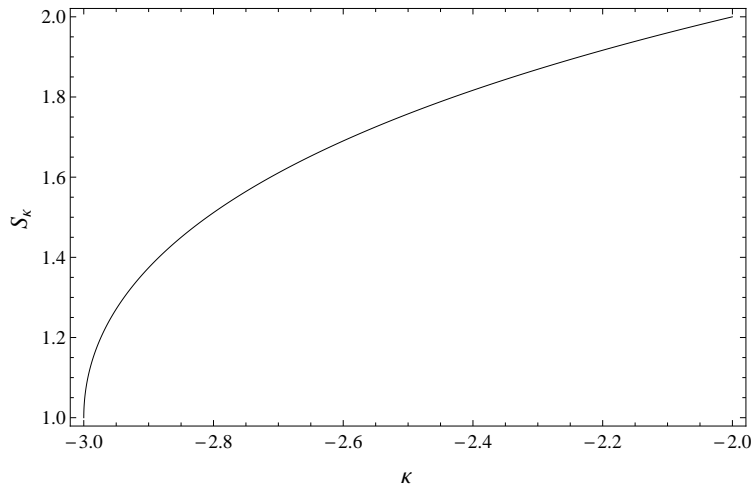

FIG. 3: Action S_κ (associated with the solution ϕ_κ) as a function of κ.

In fact, data from Fig. 2 yield estimates of S_κ with an error of one part in 10^6. We note that the above expression, on account of Eq. (13a), limits S_κ to $1 \le S_\kappa \le 2$ if $\kappa \in [-3, -2]$.

Figure 3 shows how S_κ depends on the parameter κ for $-3 \le \kappa \le -2$. We see that the action varies continuously with the parameter κ. It is worth noting that the derivative of S_κ becomes singular exactly when the physical character of the solution abruptly changes from a dyon to a monopole, at $\kappa = -3$.

IV.   ANALYTICAL EXPRESSION FOR THE SOLUTION

So far we have proved the existence of the solutions and presented them numerically. Now we make use of the mapping method introduced in [11, 12] to obtain an analytical expression for the solution defined for all $r \in [2m, \infty)$. This method is based on the fact that, if the solution exists and has no singularities in its domain, then there must be an angular sector of the complex plane containing the domain in which the function is analytic. If we compactify this region to the complex unit disk, making a transformation from z to a new variable ω, then there should be a power series expansion for the solution in ω which is convergent for $|\omega| < 1$. To employ this method, first we use the shifted variable $s = r - 2m$. Then we define a new variable related to the old one (we are considering the analytic continuation of the solution in the complex plane) by the following conformal transformation [11]

$$s \to \omega = \frac{(1 + s/R)^{1/a} - 1}{(1 + s/R)^{1/a} + 1}, \tag{16}$$

where R and a are chosen in order to avoid the first singularity of the analytic continuation of $\phi(s)$ in the complex plane. We found that the choice $R = m$ and $a = 1$ apparently gives an infinite radius of convergence for the power series (evidence for this will be presented in what follows). The inverse transformation is readily obtained

$$w \to s = R\left[\left(\frac{1 + \omega}{1 - \omega}\right)^a - 1\right]. \tag{17}$$

Using this change of variables in Eq. (5) yields

$$(\omega - 1)^2 \omega \frac{d^2\psi}{d\omega^2} + 4m\psi \frac{d\psi}{d\omega} - 2\psi = 0, \tag{18}$$

where $\psi(\omega) = \phi(s(\omega) + 2m)$. In this new variable the Charap and Duff solutions become polynomials of degree 1 and 2. The family of solutions considered here thus interpolates, in the new variable, between a straight line and a parabola.

We want to write the solutions to Eq. (18) as a power series in ω. As we have already discussed, we are interested in solutions with $\phi(2m) = \psi(0) = -\frac{1}{4m}$. Therefore, if we write $\psi(\omega) = \frac{1}{m} \sum_{n=0}^{\infty} b_n \omega^{n+\lambda}$, we have $\lambda = 0$ and $b_0 = -\frac{1}{4}$. Substituting the series expression in Eq. (18) we see that $b_1 = \frac{1}{2}$, b_2 is left free and $b_3 = 0$. The relationship between b_2 and κ is given by $b_2 = \frac{\kappa+2}{4}$. All the other coefficients b_{n+1} with $n \ge 3$ can be determined via the relation

$$b_{n+1} = \frac{2b_n\left[n(n-1)+1\right] - b_{n-1}(n-1)(n-2) - 4\sum_{q=0}^{n-1}(q+1)b_{n-q}b_{q+1}}{(n^2 - 1)}. \tag{19}$$

After evaluating the parameters b_n we can go back to the original variable r:

$$\phi_\kappa(r) = \frac{1}{m} \sum_{n=0}^{\infty} b_n^{(\kappa)} \left(1 - \frac{2m}{r}\right)^n, \tag{20}$$

where $b_n^{(\kappa)}$ depends on κ via b_2. The first few terms of (20) are displayed below:

$$\phi_\kappa(r) = -\frac{1}{4m} + \frac{1}{2m}\left(1 - \frac{2m}{r}\right) + \frac{\kappa+2}{4m}\left(1 - \frac{2m}{r}\right)^2 - \frac{\kappa^2 + 5\kappa + 6}{16m}\left(1 - \frac{2m}{r}\right)^4 +$$

$$-\frac{\kappa^2 + 5\kappa + 6}{15m}\left(1 - \frac{2m}{r}\right)^5 + \mathcal{O}\left[\left(1 - \frac{2m}{r}\right)^6\right].$$
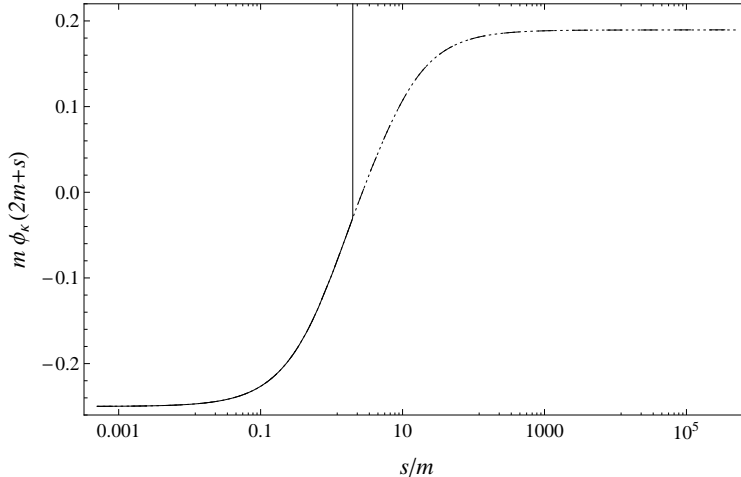

FIG. 4: Plots of the series expressions obtained before (solid line) and after (dotted line) applying the mapping method. The dashed line corresponds to the numerical solution obtained in the previous section. Notice that the series solution given by the mapping method (truncated here at n=30 000) is virtually indistinguishable from the numerical solution. In all the cases, $\kappa = -2.5$.

Figure 4 shows a typical solution obtained using this procedure and compares it with its counterparts obtained by the methods of the previous section. It illustrates the fact that the series solution whose first terms are given by Eq. (11) has a radius of convergence not greater than $2m$ and that the series solution obtained by the mapping method has an apparently infinite radius of convergence (see below). It also shows that the latter is virtually indistinguishable from the numerical solution obtained in the previous section.

Figure 5 shows the behavior of b_n for a typical value of κ. We observe numerically that $|b_n| \to 0$ as $n \to \infty$ so that $|b_n|$ is certainly limited by a constant $c > 0$. On the other hand, the series $\sum_{n=0}^{\infty} c\omega^n$ (whose sum is $c/(1-\omega)$) is absolutely convergent for $|\omega| < 1$ and using the comparison test we see that, as far as the above numerical argument is justified, the series solution will be absolutely convergent for $\omega \in [0,1)$, i.e., for any r extending from $2m$ to infinity [23].

V. CONCLUDING REMARKS

We have studied, in a systematic way, smooth spherically symmetric self-dual solutions of Yang-Mills theory in the Euclidean Schwarzschild background. We showed that our approach recovers all the previously known solutions and leads to a new one-parameter family of instantons with continuous actions in the range $1 < S < 2$. After studying the global behavior of our solutions, we exhibited them numerically and employed the mapping method [11, 12] to express them analytically.
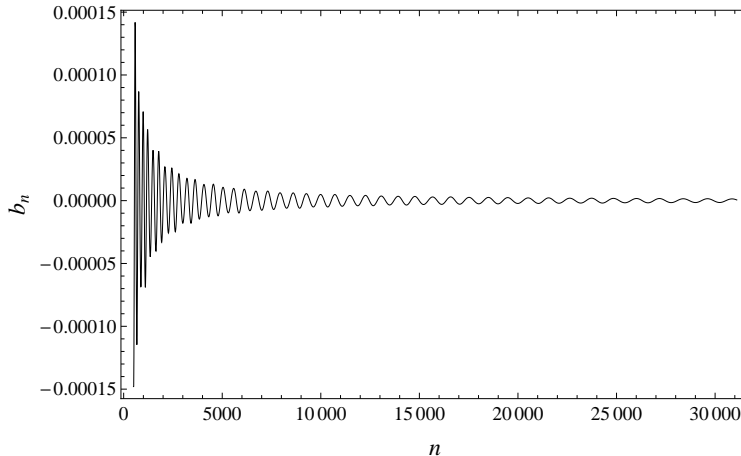
FIG. 5: Plot of the coefficients b_n as a function of n for $\kappa = -2.5$.

Finally, we examined the physical properties of this family and showed that it consists of dyons of unit (magnetic and electric) charge which interpolate between the Charap and Duff's monopole and dyon. Furthermore, the singularity in the derivative of the action S_κ with respect to the parameter κ (Fig. 3) signals the abrupt change in the physical character of the solution (from dyons to a monopole).

It is interesting to note that it has been argued, on physical grounds, that no time-dependent solutions (modulo gauge transformations) exist in Euclidean Schwarzschild space [10]. As a result, our approach would exhaust all possible smooth spherically symmetric solutions (static or not) in Schwarzschild space.

We end with a more mathematical note. Recent results impose strict constraints on the action spectrum of instantons defined on ALF geometries [13], of which the Euclidean Schwarzschild space is an example. In particular, it is possible to show that an instanton defined on the Euclidean Schwarzschild geometry will necessarily have integer action if it satisfies a certain rapidly decaying condition, which means that the curvature decays faster than $1/r^2$ as $r \to \infty$. We note that the noninteger solutions presented in this paper do not satisfy this assumption, see Eq (15), so no contradiction arises here. On the contrary, this discussion shows that the hypotheses of [13] are, in a sense, as weak as possible.

Acknowledgments

APPENDIX A: GLOBAL ANALYSIS OF THE SOLUTIONS

We show here that Eqs. (13) hold whenever $\kappa_1 < \kappa_2 \leq -2$. We start by noting that, if $\kappa < -2$, then

$$\phi'_\kappa(r) < \frac{1}{r^2} = \phi'_{-2}(r) \quad \forall r \in (2m, \infty). \tag{A1}$$

This condition is clearly true for $r - 2m \ll 1$ (see Eq. (12a)). Moreover, if this condition ceased to hold there would be a r_0 such that $\phi'_\kappa(r_0) = \frac{1}{r_0^2}$. But then, by the uniqueness theorem for ODEs, we would have $\phi_\kappa(r) = c - 1/r$ for some c which, together with the condition on $r = 2m$, leads to $\phi_\kappa(r) = \phi_{-2}(r)$ and therefore $\kappa = -2$, contradicting our assumption.

Now we prove that, if $\kappa_1 < \kappa_2 \le -2$, then

$$\phi'_{\kappa_1}(r) < \phi'_{\kappa_2}(r) \quad \forall r \in (2m, \infty). \tag{A2}$$

As already discussed, this holds for $r - 2m \ll 1$ (see Eq. (12b)). Suppose that there exists $r \in (2m, \infty)$ such that $\phi'_{\kappa_1}(r) = \phi'_{\kappa_2}(r)$ and let r_0 be the least value of r with this property. Since ϕ'_κ is continuous (due to the general theory of ODEs), we have

$$\phi'_{\kappa_1}(r) < \phi'_{\kappa_2}(r) \quad \forall r \in (2m, r_0). \tag{A3}$$

Integrating both sides of (A3) from $2m$ to $r \in (2m, r_0]$ then leads to

$$\phi_{\kappa_1}(r) < \phi_{\kappa_2}(r) \quad \forall r \in (2m, r_0], \tag{A4}$$

since $\phi_\kappa(2m) = -1/4m$ for all κ. On the other hand, Eq. (5) yields

$$\frac{1}{2}(r_0 - 2m)(\phi''_{\kappa_2} - \phi''_{\kappa_1})(r_0) + r_0 \left(\phi'_{\kappa_1}(r_0) - \frac{1}{r_0^2} \right) (\phi_{\kappa_2} - \phi_{\kappa_1})(r_0) = 0, \tag{A5}$$

since we assumed that $(\phi'_{\kappa_1} - \phi'_{\kappa_2})(r_0) = 0$. It follows from Eqs. (A1) and (A4) that $(\phi''_{\kappa_2} - \phi''_{\kappa_1})(r_0) > 0$, i.e., $\frac{d}{dr}(\phi'_{\kappa_2} - \phi'_{\kappa_1})(r_0) > 0$. In this way, $\phi'_{\kappa_2}(r_0 - \epsilon) < \phi'_{\kappa_1}(r_0 - \epsilon)$ for ϵ positive and sufficiently small. But this contradicts Eq. (A4). Therefore, no such r_0 exists and Eq. (A2) follows.

Finally, integrating Eq. (A2) from $2m$ to r we see that, for $\kappa_1 < \kappa_2 \le -2$,

$$\phi_{\kappa_1}(r) < \phi_{\kappa_2}(r) \quad \forall r \in (2m, \infty). \tag{A6}$$

This concludes the proof.

[1] A collection of classic works presenting a comprehensive perspective on the physics of instantons can be found in M. A. Shifman, "Instantons in gauge theories," *Singapore, Singapore: World Scientific (1994)*.
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[15] As will be discussed later the solutions we find are, in fact, time-dependent when expressed in the gauge where they are regular, but this presents no problem to the procedure just described.

[16] A purely imaginary α would give rise to a $\mathfrak{so}(2,1)$-valued gauge potential in (3) (see also [17]).

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[21] We note that it is not possible to start the numerical integration from $r = 2m$, since this is a singular point of the differential equation under consideration.

[22] Clearly, since (5) is a second order ODE, this property does not hold for generic solutions which do not belong to this family.

[23] It is is worth noting that the numerical approach may still be of practical relevance since the convergence rate of the series solution becomes quite slow for ω very close to 1.